SOLUTIONS OF THE EQUATIONS OF RIGID BODY DYNAMICS

(O RESHENIIAKH URAVNENII DINAMIKI TVERDOGO TELA)

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P.V. KHARLAMOV
(Novosibirsk)

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In the general case, the equations *

$$\frac{dP_1}{dt} = (P_2 + \lambda_2)\frac{\partial T}{\partial P_3} - (P_3 + \lambda_3)\frac{\partial T}{\partial P_2} + \left(\frac{\partial T}{\partial R_3} - \mu_3\right)R_2 - \left(\frac{\partial T}{\partial R_2} - \mu_2\right)R_3$$

$$\frac{dR_1}{dt} = R_2\frac{\partial T}{\partial P_3} - R_3\frac{\partial T}{\partial P_2} \qquad (123) \qquad (0.1)$$

$$2T = a_{ij}P_iP_j + b_{ij}R_iR_j + 2c_{ij}P_iR_j \qquad (0.2)$$

describe the motion of a multiply connected body in an unbounded ideal fluid. The imposition of certain conditions on the parameters

$$a_{ij},\ b_{ij},\ c_{ij},\ \lambda_i,\ \mu_i \qquad (0.3)$$

in equations (0.1) leads to much simpler problems in rigid body dynamics, such as motion about a fixed point of a body in a Newtonian central force field, the motion of a heavy gyrostat with steady internal cylindrical motion, etc. [1]. Moreover, some of the quantities λ_i and μ_i must be different from zero; otherwise, the reduction indicated simply yields the well known solution of Tisseran and Zhukovski. However, a majority of the known solutions of the general problem have been obtained precisely under the conditions

$$\lambda_i = 0, \qquad \mu_i = 0 \qquad (0.4)$$

Here, reference must be made to Chaplygin's investigations [2] on linear integrals.

Solutions with one linear integral are given in [1], where some of the restrictions of equations (0.4) have been removed. Hereinafter, solutions are obtained with two and three linear integrals. In reducing the problem to quadrature, use is also made of the following known integrals satisfying equations (0.1)

$$R_1^2 + R_2^2 + R_3^2 = R^2, \qquad (P_1 + \lambda_1)R_1 + (P_2 + \lambda_2)R_2 + (P_3 + \lambda_3)R_3 = m \qquad (0.5)$$

$$T - \mu_iR_i = h \qquad (0.6)$$

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* Translator's Note: The symbol (123) denotes that the remaining equations may be obtained by commuting subscripts.

1. Solutions with two linear integrals. Let us choose a coordinate system which is fixed in the body so that the linear integrals will be of the form

$$P_1 = k_1 R_1 + s_1 \qquad (12) \tag{1.1}$$

The constants k_1, k_2, s_1, and s_2 will be defined later.

Taking note of equations (0.1) and (1.1), the derivatives of equations (1.1) with respect to t vanish identically with respect to P_3, R_1, R_2, R_3, provided the following conditions are satisfied.

$$\begin{gathered} c_{21} = -k_1 a_{12} \qquad (12) \\ c_1 = c_3 - k_1 a_1 + (k_1 - k_2)\, a_3, \qquad c_{32} = -c_{23}, \qquad k_1 c_{23} = k_1 c_{13} = 0 \\ b_1 = b_3 + k_1 k_2 a_1 - (c_1 - c_3)(k_1 - k_2), \qquad b_{23} = 0, \qquad b_{12} = k_1 k_2 a_{12} \end{gathered} \tag{1.2}$$

$$\begin{gathered} \mu_1 = c_1 s_1 + c_{21} s_2 + c_{13}\lambda_3 - c_3 (s_1 + \lambda_1) - k_2 (a_1 s_1 + a_{12} s_2) \\ \mu_3 = c_{23} s_2 + c_{13} s_1 + c_{31} (s_1 + \lambda_1) - (c_1 + k_1 a_1)\lambda_3 \qquad (12) \end{gathered} \tag{1.3}$$

$$\begin{gathered} a_{23} = 0, \qquad (a_1 s_1 + a_{12} s_2)\,\lambda_3 = 0, \qquad (s_1 + \lambda_1)\, c_{32} = 0 \\ a_1 s_1 + a_{12} s_2 = (s_1 + \lambda_1)\, a_3 \qquad (12) \end{gathered} \tag{1.4}$$

2. First Solution. The constants k_1, k_2, s_1 and s_2 may be obtained from equations (1.2) and (1.4). For simplicity, they are assumed to be given, and the equations (1.2) and (1.4) are used to define c_{12}, c_{21}, λ_1, λ_2.

For $\lambda_3 = 0$ equations (1.2) –(1.4) are satisfied by the co-efficients of the quadratic form

$$\begin{gathered} 2T = a_1 P_1^2 + a_2 P_2^2 + a_3 P_3^2 + 2a_{12} P_1 P_2 + (b + a_1 k_1^2)\, R_1^2 + \\ + (b + a_2 k_2^2)\, R_2^2 + [b + a_3 (k_1 - k_2)^2]\, R_3^2 + 2k_1 k_2 a_{12} R_1 R_2 + \\ + 2\,[c_3 - a_1 k_1 + a_3 (k_1 - k_2)]\, P_1 R_1 + 2\,[c_3 - a_2 k_2 + a_3 (k_2 - k_1)]\, P_2 R_2 + \\ + 2c_3 P_3 R_3 - 2k_1 a_{12} P_2 R_1 - 2k_2 a_{12} P_1 R_2 \end{gathered} \tag{2.1}$$

and the relations

$$\begin{gathered} \mu_1 = [c_3 + a_3 (k_1 - k_2)]\, s_1 - \left(k_1 + k_2 + \frac{c_3}{a_3}\right)(a_1 s_1 + a_{12} s_2) \\ \lambda_1 = \left(\frac{a_1}{a_3} - 1\right) s_1 + \frac{a_{12}}{a_3}\, s_2, \qquad \mu_3 = 0 \qquad (12) \end{gathered} \tag{2.2}$$

Defining

$$J_1 = P_1 - k_1 R_1 - s_1, \qquad J_2 = P_2 - k_2 R_2 - s_2, \qquad J_3 = P_3 - (k_1 + k_2)\, R_3 \tag{2.3}$$

Equations (0.1), with the aid of equations (2.1) and (2.2), yield

$$\begin{gathered} \frac{dJ_1}{dt} = (a_3 - a_2)\, J_2 J_3 - a_{12} J_1 J_3 + 2a_3 k_1 R_3 J_2 \\ \frac{dJ_2}{dt} = -(a_3 - a_1)\, J_1 J_3 + a_{12} J_2 J_3 - 2a_3 k_2 R_3 J_1 \\ \frac{dJ_3}{dt} = (a_2 - a_1)\, J_1 J_2 + a_{12} (J_1^2 - J_2^2) + J_1 L_1 - J_2 L_2 \end{gathered} \tag{2.4}$$

$$L_1 = -2k_2 a_{12} R_1 + 2\,[a_1 k_1 + a_3 (k_2 - k_1)]\, R_2 + (a_3 - a_1)(s_2 + \lambda_2) - a_{12} (s_1 + \lambda_1) \qquad (12)$$

Assuming that

$$J_1 = 0, \qquad J_2 = 0, \qquad J_3 = \text{const} = s \tag{2.5}$$

then equations (2.4) are satisfied independently of the second group of equations (0.1). These latter may be written, in view of equations (2.5), as

$$\begin{aligned}\frac{dR_1}{dt} &= \quad 2k_1a_3R_2R_3 - a_3(s_2+\lambda_2)R_3 + a_3sR_2\\ \frac{dR_2}{dt} &= -2k_2a_3R_1R_3 + a_3(s_1+\lambda_1)R_3 - a_3sR_1\\ \frac{dR_3}{dt} &= 2a_3(k_2-k_1)R_1R_2 + a_3(s_2+\lambda_2)R_1 - a_3(s_1+\lambda_1)R_2\end{aligned} \tag{2.6}$$

Two known integrals satisfying the above are

$$R_1^2+R_2^2+R_3^2=R^2$$

$$k_1R_1^2+k_2R_2^2+(k_1+k_2)R_3^2+(s_1+\lambda_1)R_1+(s_2+\lambda_2)R_2+sR_3=m$$

and, consequently, $R_1=R_1(R_3)$, $R_2=R_2(R_3)$. The functional relationship between R_3 and t may now be determined from equations (2.6) by quadrature.

The solution thus obtained contains fourteen independent parameters

$$a_1,\ a_2,\ a_3,\ a_{12},\ b,\ k_1,\ k_2,\ c_3,\ s_1,\ s_2,\ s,\ R,\ m,\ R_3^\circ$$

The origin of coordinates will now be shifted to the mass center of the body, and the coordinate axes will be taken coincident with the principal axes of inertia (this development is similar to Sections 3 and 5, [1]). In the new coordinate system, equations (2.1), (2.2) and (2.3) are given by

$$\begin{aligned}2T &= a_1P_1^2+a_2P_2^2+a_3P_3^2+2(c_1P_1R_1+c_2P_2R_2+c_3P_3R_3)+2c_{12}(P_1R_2+P_2R_1)+\\ &+\left\{b+\frac{2a_1\,[(a_3-a_2)(c_3-c_1)+a_3(c_3-c_2)]^2+c_{12}^2\,[a_1^2(2a_2-a_3)+a_2^2a_3]}{2\,[a_3(a_1+a_2)-a_1a_2]^2}\right\}R_1^2+\\ &+\left\{b+\frac{2a_2\,[(a_3-a_1)(c_3-c_2)+a_3(c_3-c_1)]^2+c_{12}^2\,[a_2^2(2a_1-a_3)+a_1^2a_3]}{2\,[a_3(a_1+a_2)-a_1a_2]^2}\right\}R_2^2+\\ &+\left\{b+a_3\frac{[a_1(c_3-c_2)-a_2(c_3-c_1)]^2+c_{12}^2(a_1+a_2)^2}{[a_3(a_1+a_2)-a_1a_2]^2}\right\}R_3^2+\\ &+2c_{12}\frac{a_2(c_3-c_1)+a_1(c_3-c_2)}{a_3(a_1+a_2)-a_1a_2}R_1R_2\end{aligned}$$

$$\mu_1=\left[c_1a_3-c_3a_1-a_1a_3\frac{(a_3-a_1)(c_3-c_2)+a_3(c_3-c_1)}{a_3(a_1+a_2)-a_1a_2}\right]\frac{\lambda_1}{a_2-a_3}+ \frac{2a_3^2(a_1+a_2)c_{12}}{a_3(a_1+a_2)-a_1a_2}\,\frac{\lambda_2}{a_2-a}$$

$$\mu_2=\left[c_2a_3-c_3a_2-a_2a_3\frac{(a_3-a_2)(c_3-c_1)+a_3(c_3-c_2)}{a_3(a_1+a_2)-a_1a_2}\right]\frac{\lambda_2}{a_2-a_3}+ \frac{2a_3^2(a_1+a_2)c_{12}}{a_3(a_1+a_2)-a_1a_2}\,\frac{\lambda_1}{a_1-a_3}, \qquad \lambda_3=0,\quad \mu_3=0$$

$$P_1=\frac{(a_3-a_2)(c_3-c_1)+a_3(c_3-c_2)}{a_3(a_1+a_2)-a_1a_2}R_1+\frac{a_2c_{12}}{a_3(a_1+a_2)-a_1a_2}R_2+\frac{a_3}{a_1-a_3}\lambda_1$$

$$P_2=\frac{(a_3-a_1)(c_3-c_2)+a_3(c_3-c_1)}{a_3(a_1+a_2)-a_1a_2}R_2+\frac{a_1c_{12}}{a_3(a_1+a_2)-a_1a_2}R_1+\frac{a_3}{a_2-a_3}\lambda_2$$

$$P_3=\frac{(2a_3-a_2)(c_3-c_1)+(2a_3-a_1)(c_3-c_2)}{a_3(a_1+a_2)-a_1a_2}R_3+s$$

From equations (2.4) it is easy to obtain two solutions with arbitrary initial data, generalizing the cases of integrability due to Steklov [3] and Liapunov [4]. Thus

$$\begin{aligned}\frac{d}{dt}\{(a_3-a_1)J_1^2+(a_3-a_2)J_2^2\} &= 2a_{12}\{(a_3-a_2)J_2^2-(a_3-a_1)J_1^2\}J_3+\\ &+4a_3\{k_1(a_3-a_1)-k_2(a_3-a_2)\}R_3J_1J_2\end{aligned}$$

Let $a_{12} = 0$, $k_1 = \varkappa (a_3 - a_2)$, $k_2 = \varkappa (a_3 - a_1)$. Then

$$(a_3 - a_1) J_1^2 + (a_3 - a_2) J_2^2 = \text{const}$$

or, taking into account equations (2.3) and (2.2),

$$(a_3 - a_1)\left[P_1 - \varkappa (a_3 - a_2) R_1 + \frac{a_3}{a_3 - a_1} \lambda_1\right]^2 +$$
$$+ (a_3 - a_2)\left[P_2 - \varkappa (a_3 - a_1) R_2 + \frac{a_3}{a_3 - a_2} \lambda_2\right]^2 = \text{const}$$

Now, if $a_1 = a_2 = a_3 = a$, $a_{12} = 0$, then equations (2.4) yield

$$k_1 J_1^2 + k_2 J_2^2 = \text{const}$$

From equations (1.3) and (2.2),

$$k_1 = \frac{c_3 - c_2}{a}, \qquad k_2 = \frac{c_3 - c_1}{a}, \qquad s_1 = \frac{\mu_1}{2 (c_1 - c_3)}, \qquad s_2 = \frac{\mu_2}{2 (c_2 - c_3)}$$

and, consequently,

$$(c_1 - c_3)\left[P_1 + \frac{c_2 - c_3}{a} R_1 - \frac{\mu_1}{2 (c_1 - c_3)}\right]^2 +$$
$$+ (c_2 - c_3)\left[P_2 + \frac{c_1 - c_3}{a} R_2 - \frac{\mu_2}{2 (c_2 - c_3)}\right]^2 = \text{const}$$

These cases of integrability were obtained in [1] using a different approach.

3. Second Solution. We now let $\lambda_3 \neq 0$ and restrict consideration to cases where $k_1 = k_2 = 0$. Using equations (1.2) – (1.4) to determine s_1, s_2 and the constants in expression (0.3), leads to

$$2T = a_1 P_1^2 + a_2 P_2^2 + a_3 P_3^2 + b (R_1^2 + R_2^2 + R_3^2) + 2c (P_1 R_1 + P_2 R_2 + P_3 R_3) +$$
$$+ 2c_{23} (P_2 R_3 - P_3 R_2) + 2c_{13} (P_1 R_3 - P_3 R_1)$$

$$\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = \lambda, \quad \mu_1 = c_{13}\lambda, \quad \mu_2 = c_{23}\lambda, \quad \mu_3 = - c\lambda, \quad P_1 = 0, \quad P_2 = 0$$

The integrals in equations (0.5) and (0.6), which in this case may be written as

$$R_1^2 + R_2^2 + R_3^2 = R^2, \qquad (P_3 + \lambda) R_3 = m$$

$$2 (P_3 + \lambda) (c_{13} R_1 + c_{23} R_2) = a_3 P_3^2 + 2c (P_3 + \lambda) R_3 + bR^2 - 2h$$

then yield P_3, R_1 and R_2 as functions of R_3, and the last of equations (0.1) now takes the form

$$\left(\frac{dR_3}{dt}\right)^2 = (c_{13}^2 + c_{23}^2) (R^2 - R_3^2) R_3^2 - \frac{1}{4m^2} \left[a_3 (m - \lambda R_3)^2 + (2cm + bR^2 - 2h) R_3^2\right]$$

and consequently R_3 is an elliptic function of time.

If the origin of coordinates is shifted to the mass center of the body, the solution may be written as

$$2T = a_1 P_1^2 + a_2 P_2^2 + a_3 P_3^2 + \left[b - 4 \frac{a_1 c_{13}^2}{(a_1 - a_3)^2}\right] R_1^2 + \left[b - 4 \frac{a_2 c_{23}^2}{(a_2 - a_3)^2}\right] R_2^2 +$$
$$+ \left[b - 4 \frac{a_3 c_{13}^2}{(a_1 - a_3)^2} - 4 \frac{a_3 c_{23}^2}{(a_2 - a_3)^2}\right] R_3^2 - 4 \frac{a_1 + a_2}{(a_1 - a_3)(a_2 - a_3)} c_{13} c_{23} R_1 R_2 +$$
$$+ 2c (P_1 R_1 + P_2 R_2 + P_3 R_3) + 2c_{13} (P_1 R_3 + P_3 R_1) + 2c_{23} (P_2 R_3 + P_3 R_2)$$

$$\lambda_1 = \lambda_2 = 0, \quad \lambda_3 = \lambda, \quad \mu_1 = \frac{a_1 + a_3}{a_3 - a_1} c_{13}\lambda, \quad \mu_2 = \frac{a_2 + a_3}{a_3 - a_2} c_{23}\lambda, \quad \mu_3 = - c\lambda$$

$$P_1 + \frac{2c_{13}}{a_1 - a_3} R_1 = 0, \qquad P_2 + \frac{2c_{23}}{a_2 - a_3} R_2 = 0$$

4. Solutions with three linear integrals. By the same procedure as used in Section 1, the conditions for the existence of a set of integrals

$$P_1 = k_1 R_1 + s_1 \qquad (123) \tag{4.1}$$

take the form

$$\begin{gathered} b_2 - b_3 = (k_2 + k_3 - k_1)(c_3 - c_2) + (k_1 - k_2) k_3 a_3 - (k_3 - k_1) k_2 a_2 \\ b_{23} = -k_2 c_{23} = -k_3 c_{32}, \qquad (k_3 - k_2)(c_{13} + k_3 a_{31}) = 0 \\ (k_2 - k_3)(c_{12} + k_2 a_{12}) = 0 \\ \nu_2 \alpha_3 - \nu_3 \alpha_2 = 0, \qquad (c_{31} + k_1 a_{31}) \nu_2 - (c_{21} + k_1 a_{12}) \nu_3 = 0 \\ \beta_1 = (k_3 - k_2) \alpha_1 + (c_2 + k_2 a_2) \nu_1 - (c_{12} + k_2 a_{12}) \nu_2 \\ \beta_1 = (k_2 - k_3) \alpha_1 + (c_3 + k_3 a_3) \nu_1 - (c_{13} + k_3 a_{31}) \nu_3 \\ (123) \end{gathered} \tag{4.2}$$

Here

$$\begin{gathered} \alpha_1 = a_1 s_1 + a_{12} s_2 + a_{31} s_3, \qquad \beta_1 = c_1 s_1 + c_{21} s_2 + c_{31} s_3 - \mu_1 \\ \nu_1 = s_1 + \lambda_1 \qquad (123) \end{gathered} \tag{4.3}$$

Chaplygin [2], under conditions given by equations (0.4), confined himself to the analysis of cases in which

$$(k_2 - k_3)(k_3 - k_1)(k_1 - k_2) \neq 0 \tag{4.4}$$

or

$$k_1 = k_2 = k_3 = 0 \tag{4.5}$$

But for these cases it is also possible to remove some of the restrictions of equations (0.4). Thus, in case condition (4.4) holds, instead of equations (0.4), it is sufficient to require that the following conditions be satisfied:

$$\begin{gathered} (a_1 - a) s_1 + a_{12} s_2 + a_{31} s_3 = a \lambda_1 \\ \mu_1 + [c + a (k_1 + k_2 + k_3)] \lambda_1 = a (k_1 - k_2 - k_3) s_1 \qquad (123) \end{gathered}$$

(Chaplygin denoted the parameters c and a by μ and $-\frac{1}{2}\lambda$, respectively.) If equations (4.5) hold, the corresponding conditions are

$$s_1 = -\lambda_1, \qquad \mu_1 = -c_1 \lambda_1 - c_{21} \lambda_2 - c_{31} \lambda_3 \qquad (123)$$

Note also that, for $k_1 = k_2 = k_3 = k \neq 0$ equations (4.2) are satisfied by the coefficients of the quadratic form

$$\begin{aligned} 2T = {} & a_1 P_1^2 + a_2 P_2^2 + a_3 P_3^2 + 2 (a_{23} P_2 P_3 + a_{31} P_3 P_1 + a_{12} P_1 P_2) + \\ & + b_1 R_1^2 + \frac{(c_1 - c_2) b_3 - (c_3 - c_2) b_1}{c_1 - c_3} R_2^2 + b_3 R_3^2 + 2 (c_1 P_1 R_1 + c_2 P_2 R_2 + c_3 P_3 R_3) \end{aligned}$$

provided that

$$k = -\frac{b_1 - b_3}{c_1 - c_3}, \qquad s_1 = -\lambda_1, \qquad \mu_1 = -c_1 \lambda_1 \qquad (123)$$

In the following sections, two more solutions are given for the case $k_1 = k_3 \neq k_2$, not investigated by Chaplygin.

5. Third Solution. Assume $k_2 = 0$. Equations (4.2) are satisfied by the coefficients of the quadratic form

$$2T = a_1P_1^2 + a_2P_2^2 + a_3P_3^2 + 2a_{23}P_2P_3 + 2a_{31}P_3P_1 + \left[b + a_1\left(\frac{c_1 - c_3}{a_1 - a_3}\right)^2\right]R_1^2 + bR_2^2 + \left[b + a_3\left(\frac{c_1 - c_3}{a_1 - a_3}\right)^2\right]R_3^2 + 2\,(c_1P_1R_1 + c_2P_2R_2 + c_3P_3R_3 + c_{21}P_2R_1 + c_{23}P_2R_3)$$

and the relations

$$s_1 = -\frac{a_{12}}{a_1}s, \quad s_2 = s, \quad s_3 = -\frac{a_{23}}{a_3}s, \quad \lambda_1 = \frac{a_{12}}{a_1}s, \quad \lambda_3 = \frac{a_{23}}{a_3}s$$

$$\mu_1 = \left(c_{21} - c_1\frac{a_{12}}{a_1}\right)s, \quad \mu_2 = c_2 s + \frac{c_1a_3 - c_3a_1}{a_1 - a_3}(s + \lambda_2), \quad \left(\mu_3 = c_{23} - c_3\frac{a_{23}}{a_3}\right)s$$

In addition,

$$P_1 + \frac{c_1 - c_3}{a_1 - a_3}R_0\cos\varphi + \frac{a_{12}}{a_1}s = 0, \quad P_2 = s, \quad P_3 + \frac{c_1 - c_3}{a_1 - a_3}R_0\sin\varphi + \frac{a_{23}}{a_3}s = 0$$

$$R_1 = R_0\cos\varphi, \qquad R_2 = R_2^\circ, \qquad R_3 = R_0\sin\varphi$$

where φ is an elementary function of time given by

$$t = \int_{\varphi_0}^{\varphi}\left[\omega_0 + \left(c_{23} - \frac{c_1 - c_3}{a_1 - a_3}a_{23}\right)R_0\sin\varphi + \left(c_{21} - \frac{c_1 - c_3}{a_1 - a_3}a_{12}\right)R_0\cos\varphi\right]^{-1}d\varphi$$

$$\omega_0 = \left(a_2 - \frac{a_{12}^2}{a_1} - \frac{a_{23}^2}{a_3}\right)s + \left(c_2 + \frac{c_1a_3 - c_3a_1}{a_1 - a_3}\right)R_2^\circ$$

This solution contains the sixteen parameters a_1, a_2, a_3, a_{23}, a_{12}, b_2, c_1, c_2, c_3, c_{23}, c_{21}, λ_2, s, R_0, R_2°, φ_0.

6. Fourth Solution. From equations (4.2), the coefficients in the quadratic forms, equations (0.2) are found to be

$$b_1 = b_2 + n\,(c_2 - c) + k^2a_1, \qquad b_{12} = kna_{12}, \qquad b_{13} = 0, \qquad a_{13} = 0$$

$$c_{12} = -na_{12}, \qquad c_{21} = -k_1a_{21}, \qquad c_{31} = c_{13} = 0 \quad (13) \qquad (6.1)$$

where (6.2)

$$k = -\frac{c_1 - c_3}{a_1 - a_3}, \quad n = \frac{2a\,(c_1 - c_3) + a_1\,(c_3 - c_2) - a_3\,(c_1 - c_3)}{(2a - a_2)\,(a_1 - a_3)}, \quad c = \frac{c_3a_1 - c_1a_3}{a_1 - a_3}$$

The parameter a is arbitrary. In addition, the following are obtained

$$s_1 = a\frac{[(a_2 - a)\,(a_3 - a) - a_{23}^2]\,\lambda_1 - (a_3 - a)\,a_{12}\lambda_2 + a_{12}a_{23}\lambda_3}{(a_1 - a)\,(a_2 - a)\,(a_3 - a) - (a_1 - a)\,a_{23}^2 - (a_3 - a)\,a_{12}^2}$$

$$s_2 = a\frac{(a_1 - a)\,(a_3 - a)\,\lambda_2 - (a_1 - a)\,a_{23}\lambda_3 - (a_3 - a)\,a_{12}\lambda_1}{(a_1 - a)\,(a_2 - a)\,(a_3 - a) - (a_1 - a)\,a_{23}^2 - (a_3 - a)\,a_{12}^2} \qquad (6.3)$$

$$s_3 = a\frac{[(a_2 - a)\,(a_1 - a) - a_{12}^2]\,\lambda_3 - (a_1 - a)\,a_{23}\lambda_2 + a_{12}a_{23}\lambda_1}{(a_1 - a)\,(a_2 - a)\,(a_3 - a) - (a_1 - a)\,a_{23}^2 - (a_3 - a)\,a_{12}^2}$$

$$\mu_1 = -c\lambda_1 - an\nu_1, \qquad \mu_2 = -c\lambda_2 - an\nu_2 - 2aks_2, \qquad \mu_3 = -c\lambda_3 - an\nu_3 \qquad (6.4)$$

Under these conditions, equations (0.1) are satisfied by the set of integrals

$$P_1 = kR_1 + s_1, \qquad P_2 = nR_2 + s_2, \qquad P_3 = kR_3 + s_3 \qquad (6.5)$$

The constants k, n and s_i are determined from equations (6.2) and (6.3).

Taking into account equations (6.5) and (4.2), the integrals, equations (0.5), may be written as

$$R_1^2 + R_3^2 = R^2 - R_2^2, \qquad \nu_1R_1 + \nu_2R_2 = m - kR^2 - \nu_2R_2 - (n - k)\,R_2^2$$

Hence

$$(\nu_1^2 + \nu_3^2) R_1 = \nu_1 [m - kR^2 - \nu_2 R_2 - (n-k) R_2^2] - \\ - \nu_3 \sqrt{(\nu_1^2 + \nu_3^2)(R^2 - R_2^2) - [m - kR^2 - \nu_2 R_2 - (n-k) R_2^2]^2} \tag{6.6}$$

$$(\nu_1^2 + \nu_3^2) R_3 = \nu_3 [m - kR^2 - \nu_2 R_2 - (n-k) R_2^2] + \\ + \nu_1 \sqrt{(\nu_1^2 + \nu_3^2)(R^2 - R_2^2) - [m - kR^2 - \nu_2 R_2 - (n-k) R_2^2]^2}$$

One of equations (0.1), namely

$$\frac{dR_2}{dt} = R_3 \frac{\partial T}{\partial P_1} - R_1 \frac{\partial T}{\partial P_3}$$

combined with equations (0.2), (6.1), (6.5) and (6.6) now determines R_2 as an elliptic function of time

$$at = \int_{R_2^\circ}^{R_2} \{(\nu_1^2 + \nu_3^2)(R^2 - R_2^2) - [m - kR^2 - \nu_2 R_2 - (n-k) R_2^2]^2\}^{-1/2} dR_2$$

Moreover, equations (6.5) and (6.6) now determine the remaining variables as functions of time.

The solution thus obtained contains the sixteen parameters

$$a_1, a_2, a_3, a_{12}, a_{23}, b_2, c_1, c_2, c_3, \lambda_1, \lambda_2, \lambda_3, a, m, R, R_2^\circ \tag{6.7}$$

It is remarkable by its relation to some solutions of the classical problems concerning the motion of a heavy body about a fixed point.

The parameters listed in (6.7) are now subjected to the additional conditions

$$a_{12} = a_{23} = 0, \quad a = {}^1/_2 a_2, \quad b_2 = 0, \quad c_1 = c_2 = c_3 = 0$$

Hence, from equations (6.3) and (6.4),

$$\mu_1 = \frac{a_1 a_2}{a_2 - 2a_1} n\lambda_1, \quad \mu_2 = 0, \quad \mu_3 = \frac{a_3 a_2}{a_2 - 2a_3} n\lambda_3$$

$$s_1 = \frac{a_2}{2a_1 - a_2}\lambda_1, \quad s_2 = \lambda_2, \quad s_3 = \frac{a_2}{2a_3 - a_2}\lambda_3 \tag{6.8}$$

The integrals, equations (6.5), may now be written as

$$P_1 = \frac{a_2}{2a_1 - a_2}\lambda_1, \quad P_2 = nR_2 + \lambda_2, \quad P_3 = \frac{a_2}{2a_3 - a_2}\lambda_3 \tag{6.9}$$

If the quantities

$$a_1, a_2, a_3; \quad P_1, P_2, P_3; \quad \frac{\mu_1}{n}, \frac{\mu_3}{n}; \quad nR_2$$

are, respectively, defined as

$$\frac{1}{A}, \frac{1}{B}, \frac{1}{C}, \quad Ap, Bq, Cr; \quad -\nu\cos\alpha, -\nu\sin\alpha; \quad -\frac{\gamma_2}{\nu}$$

Equations (6.8) and (6.9) take the forms

$$\lambda_1 = (2B - A)\nu\cos\alpha, \quad \lambda_3 = (2B - C)\nu\sin\alpha$$

$$p = \frac{\lambda_1}{2B - A} = \nu\cos\alpha, \quad \gamma_2 = \nu(\lambda_2 - Bq), \quad r = \frac{\lambda_3}{2B - C} = \nu\sin\alpha$$

These conditions characterize the cases of integrability given in [5], which include the known Bobylev [6] – Steklov [7] solution.

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